just as in 1°, saddle points. In this case the sum of the indices of the singularities lying at the boundary is different from zero.

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THE MOTION OF A SYSTEM OF VORTEX RINGS IN AN INCOMPRESSIBLE FLUID*

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Hamiltonian formalism is developed in the problem of the motion of a system of coaxial vortex rings in an infinite, incompressible ideal fluid. An additional invariant of the motion representing the momentum of the surrounding fluid, is determined. In the case of two vortex rings the equations of motion are found to be completely integrable, and this explains the mutual slip-through of the vortex rings described qualitatively by Helmholtz. The influence of viscosity on the initial stage of motion is assessed.

1. Hamiltonian formulation. The problem of the motion of vortex rings which has already been studied in the last century, represents the simplest case of a three-dimensional vortex flow. Even in this simplest case a theoretical analysis is possible only when the radius of the vortex ring is much greater than the radius of the vortex core. Let us consider a system of coaxial vortex rings moving through an infinite, ideal incompressible fluid at rest at infinity. We shall introduce a cylindrical r, z, θ-coordinate system where the z axis is directed along the general axis of the vortex rings. Let Γ_α be the circulation of the vortex ring with index $\alpha, \alpha = 1, \ldots, N$, R_α be the ring radius, a_α the radius of the vortex core and z_α the longitudinal coordinate of the vortex. We shall seek the veclocity field outside the vortex rings in the form

$$\mathbf{v} = \operatorname{rot} \mathbf{A} \tag{1.1}$$

Symmetry considerations imply that $\mathbf{A} = A(r, z)\mathbf{e}_\theta$. Then from (1.1) we obtain

$$v_r = -\frac{\partial A}{\partial z}, \quad v_z = \frac{1}{r}\frac{\partial (rA)}{\partial r} \tag{1.2}$$

Substituting (1.2) into the equation

$$\operatorname{rot} \mathbf{v} = \sum_{\alpha=1}^{N} \Gamma_\alpha \delta(r - R_\alpha)\,\delta(z - z_\alpha)$$

we arrive at the following equation for the vector potential A:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} = -\sum_{\alpha=1}^{N} \Gamma_\alpha \delta(r - R_\alpha)\,\delta(z - z_\alpha) \tag{1.3}$$

The right-hand side of (1.3) is obtained under the assumption that $a/R \ll 1$, and expresses the fact that the circulation along any closed contour enclosing the vortex with index α is equal to Γ_α. Since (1.3) is linear, it follows that the solution can be expressed as the sum of solutions for the isolated vortex rings, and has the form

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$$A(r, z) = \frac{1}{2} \sum_{\alpha=1}^{N} \Gamma_\alpha R_\alpha \int_0^\infty J_1(kR_\alpha) J_1(kr) e^{-k|z-z_\alpha|} dk \tag{1.4}$$

where J_1 is the Bessel function. Substituting (1.4) into (1.2) we obtain expressions for the components of the velocity of motion of the vortices

$$\frac{dR_\alpha}{dt} = \frac{1}{2} \sum_{\beta=1}^{N}{}' \Gamma_\beta R_\beta B_{\alpha\beta} \operatorname{sgn}(z_\alpha - z_\beta) \tag{1.5}$$

$$\frac{dz_\alpha}{dt} = \frac{dz_\alpha{}^\circ}{dt} + \frac{1}{2} \sum_{\beta=1}^{N}{}' \Gamma_\beta R_\beta C_{\alpha\beta} + \frac{1}{2R_\alpha} \sum_{\beta=1}^{N}{}' \Gamma_\beta R_\beta D_{\alpha\beta} \tag{1.6}$$

where

$$B_{\alpha\beta} = \int_0^\infty kJ_1(kR_\alpha) J_1(kR_\beta) e^{-k|z_\alpha - z_\beta|} dk$$

$$C_{\alpha\beta} = \int_0^\infty kJ_1(kR_\beta) J_1'(kR_\alpha) e^{-k|z_\alpha - z_\beta|} dk$$

$$D_{\alpha\beta} = \int_0^\infty J_1(kR_\alpha) J_1(kR_\beta) e^{-k|z_\alpha - z_\beta|} /dR$$

A prime accompanying the summation sign means that a term with index α is omitted. The expressions for $B_{\alpha\beta}$, $C_{\alpha\beta}$, $D_{\alpha\beta}$ can be written in terms of elliptic integrals. All terms in (1.5) and (1.6), apart from $dz_\alpha{}^\circ/dt$, are related to the action excerted on the vortex with index α by all the remaining vortices.

When formal differentiation is carried out, unusual features appear in (1.2) connected with the action of a vortex ring on itself. As we know, their removal requires the introduction of a model of a vortex core. As a result, we obtain the following expression for the selfinduced velocity of a vortex ring apart from terms of order $O[(a/R^2)\ln(R/a)]$, which we neglect by virtue of the initial assumption that $a/R \ll 1$:

$$\frac{dz_\alpha{}^\circ}{dt} = \frac{\Gamma_\alpha}{4\pi R_\alpha} \left[\ln\left(\frac{8R_\alpha}{a_\alpha}\right) - E \right]$$

The constant E depends on the form of the distribution of the vorticity within the vortex ring (details are given in /1/).

For example, in the case of homogeneous vorticity (Kelvin's model) we have $E = 0.25$, and in the case of a hallow vortex (Hicks's model) we have $E = 0.5$. Kelvin's theorem on circulation implies that the quantities R and a interdependent: in the first case $Ra^2 = \text{const}$, and in the second case $Ra = \text{const}$. Therefore, the motion of the system of N vortices depends only on $2N$ variables z_α and R_α is independent of a_α.

Let us change to the Hamiltonian formulation of (1.5) and (1.6). We obtain the first integral from (1.5)

$$P = \sum_{\alpha=1}^{N} \Gamma_\alpha R_\alpha{}^2 = \text{const} \tag{1.7}$$

We know that $P = \Gamma R^2$ is the momentum of the fluid surrounding the vortex ring, apart from a constant multiplier. Equation (1.7) expresses the law of conservation of momentum of the system in question.

We can naturally write (1.5) in the form

$$\Gamma_\alpha \frac{dR_\alpha{}^2}{dt} = -\frac{\partial H}{\partial z_\alpha} \tag{1.8}$$

Indeed, the initial system of vortex rings is invariant with respect to displacement along the z axis and, provided that the Hamiltonian H also has this property, (1.8) will automatically yield the invariant (1.7). Equating (1.5) and (1.8) we obtain

$$H = \frac{1}{2} \sum_{\alpha, \beta=1}^{N}{}' \Gamma_\alpha \Gamma_\beta R_\alpha R_\beta D_{\alpha\beta} + h(R_1, \ldots, R_N) \tag{1.9}$$

By analogy with Kirchhoff's equations /2/ of the motion of discrete plane vortices, we will write the equation (1.6) in the form

$$\Gamma_\alpha \frac{dz_\alpha}{dt} = \frac{\partial H}{\partial R_\alpha{}^2} \tag{1.10}$$

Equating (1.6) to (1.10) we obtain

$$h = \sum_{\alpha=1}^{N} h_\alpha(R_\alpha), \quad \frac{dh_\alpha}{dR_\alpha} = \frac{\Gamma_\alpha^2}{2\pi}\left[\ln\left(\frac{8R_\alpha}{a_\alpha}\right) - E\right] \tag{1.11}$$

Thus the motion of a system of vortex rings can be represented in Hamiltonian form by (1.8), (1.10) with Hamiltonian (1.9), (1.11). As expected, the Hamiltonian H is invariant with respect to displacement along the z axis, and the system of vortex rings in question has, in addition to the invariant $I_1 = H$, usual for the Hamiltonian systems, the invariant $I_2 = P$. We also note that the equations (1.8), (1.10), like the classical Kirchhoff equations, are not written in canonical form. To reduce the equations (1.8), (1.10) to canonical form it is sufficient to introduce the generalised coordinates $q_\alpha = z_\alpha$ and moments $p_\alpha = \Gamma_\alpha R_\alpha^2$. The Hamiltonian character of (1.8), (1.10) in physical coordinates can also be shown by introducing the Poisson brackets

$$\{f, g\} = \sum_{\alpha=1}^{N} \frac{1}{\Gamma_\alpha}\left(\frac{\partial f}{\partial z_\alpha}\frac{\partial g}{\partial R_\alpha^2} - \frac{\partial f}{\partial R_\alpha^2}\frac{\partial g}{\partial z_\alpha}\right)$$

2. Examples of the motion of the vortex rings. We shall study in more detail a system of double vortex rings. The systems has two degrees of freedom, two first integrals, and is fully integrable by virtue of Liouville's theorem. From the topological point of view the surface of the level of the integrals of motion $H = \text{const}, P = \text{const}$ is equivalent to a plane, cylinder or torus /3/. For the first case we have e.g. the corresponding motion of vortex rings in which the distance separating them tends to infinity as $t \to \infty$. For the second case we have a slip-through sequence of vortex rings already described analytically by Helmholz, and motions corresponding to the third case are obviously impossible.

Although Liouville's theorem guarantees the full integrability of the equations of motion of a system of two vortex rings, the authors did not succeed in expressing the solution in quadratures. We shall therefore consider slipping through of two identical vortex rings of equal intensity, under the assumption that

$$l = [(R_1 - R_2)^2 + (z_1 - z_2)^2]^{1/2} \ll (R_1 + R_2)/2$$

To make the models of vortex rings usable we must also assume that $l \gg a$. It is clear from the analytical point of view that under these assumptions the interaction between the vortex rings is analogous to the interaction between the rectilinear vortex filaments. This follows from the formal expansion of the Hamiltonian (1.9), (1.11) apart from terms of order $O(l/R)$, which will be neglected

$$H = -\frac{\Gamma^2\sqrt{R_1R_2}}{\pi}\ln\frac{l}{a} + \frac{\Gamma^2\sqrt{R_1R_2}}{\pi}\left[\ln\frac{4(R_1+R_2)}{a} - 2\right] + h_1(R_1) + h_2(R_2) \tag{2.1}$$

Indeed, the first term of (2.1) is a Hamiltonian of the interaction between two vortex filaments. It determines the dynamic of motion of the vortices, since within the approximation used the influence of the remaining terms is reduced to the effect of displacing the vortices with constant velocity along the z axis.

The period $T = 4\pi^2 l^2/\Gamma$ of slipping through can be found from (2.1). Within a single period the system will move along the z axis by a distance

$$L = \frac{\pi l^2}{R}\left(\ln\frac{8R}{a} + \ln\frac{8R}{l} - E\right)$$

Analogous arguments can be carried out for a system of N vortex rings of equal intensity, distributed near each other ($a \ll l \ll R$). In this case the motion of vortex rings is determined, as before, by the interaction between the corresponding vortex filaments which are displaced with constant velocity along the z axis. It follows that the known solutions for the rectilinear vortex filaments can be used to obtain the solutions for the vortex rings. In particular, we know that a system of N equal vortex filaments distributed along a single straight line at a constant distance from each other, rotates about the centre of vorticity with constant angular velocity. In the case of vortex rings the solution corresponds to the slipping through of N vortex rings. We note that the mutual slipping through of two vortex rings has been confirmed experimentally /4, 5/.

3. Effect of low viscosity. In the case of a viscous incompressible fluid and motion of even a single vortex ring is unsteady. The problem was studied in detail in /6/ under the assumption that the characteristic distance by which the vorticity has diffused is much smaller than the ring radius, i.e. $\nu t \ll R^2$ where ν is the coefficient of kinematic viscosity. Here it was assumed that at the initial instant all the vorticity was concentrated on the circle of radius R. The velocity of motion of a single vortex ring is given, to terms of order

$$O\left[\left(\frac{\nu t}{R^2}\right)^{1/2}\ln\frac{\nu t}{R^2}\right]$$

which are neglected, by the formula /6/

$$\frac{dz^0}{dt}=\frac{\Gamma}{4\pi R}\left[\ln\left(\frac{8R}{\sqrt{4\nu t}}\right)-E\right]$$

where $E\approx 0.558$ and Γ is a constant within the approximation used (in fact, Γ can be assumed constant to within exponentially small terms $O(\exp(-R^2/4\nu t))$).

Using the same assumptions, we shall now consider the problem of the motion of a system of N coaxial vortex rings in an infinite, incompressible viscous fluid. Combining the results of /6/ with those of Sect.1, we arrive at the following conclusions:

the equations of motion can be reduced, as before, to the Hamiltonian form (1.8), (1.10);

the Hamiltonian of the system can again be represented in the form (1.9) where

$$h=\sum_{\alpha=1}^{N}h_\alpha(R_\alpha,t),\quad \frac{dh_\alpha}{dR_\alpha}=\frac{\Gamma_\alpha^2}{2\pi}\left[\ln\left(\frac{8R_\alpha}{\sqrt{4\nu t}}\right)-0.558\right] \tag{3.1}$$

we note that when the fluid is viscous, the Hamiltonian depends explicitly on time and is not an invariant of the motion;

The momentum P of the system of vortices (1.7) is, as before, an invariant of the motion.

We can write the expression for the Hamiltonian (1.9), (3.1) in the form

$$H=H_0-\frac{1}{4\pi}\ln(4\nu t)\sum_{\alpha=1}^{N}\Gamma_\alpha^2R_\alpha$$

$$H_0=\frac{1}{2}\sum_{\alpha,\beta=1}^{N}{}'\,\Gamma_\alpha\Gamma_\beta R_\alpha R_\beta D_{\alpha\beta}+\sum_{\alpha=1}^{N}\frac{\Gamma_\alpha^2}{2\pi}R_\alpha[\ln(8R_\alpha)-1.558]$$

and here H_0 does not depend explicitly on time.

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